Math 221-Midterm 1

Name:________________________________

Your TA: (circle one) Your discussion session time:____________

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Directions–Please Read Carefully!

- **There are 6 problems and 8 pages.**
- **No calculators/electronic devices. Cellphones OFF. No outside resources. No scratch paper, you have enough space to show your work.**
- **Show all your work. Justify your answers and write clearly. Unsubstantiated/illegible answers will receive little or no credit.**
- **If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of this on the problem page so that the grader knows where to find your answers. Raise your hand if more scratch paper is needed or if you have any questions.**
- ***Good Luck!!!***

Problem Number	Possible Points	Score
1	15	
2	20	
3	13	
4	18	
5	16	
6	18	
Total	100	

1. (15 points) Calculate the derivatives of the following functions. **You do NOT need to simplify your answer.**

(a) $f(x) = \dfrac{x^3 - 2x}{x+3}$

$$f'(x) = \frac{(3x^2-2)(x+3) - (x^3-2x)}{(x+3)^2}$$

(b) $f(x) = \left(\sqrt{x^2-1}+1\right)^{10}$

$$f'(x) = 10\left(\sqrt{x^2-1}+1\right)^9 \cdot \frac{1}{2}(x^2-1)^{-\frac{1}{2}} \cdot 2x$$

(c) $f(x) = \cos(x^2)\sin(\sqrt{x+1})$

$$f'(x) = \frac{d}{dx}\cos(x^2) \cdot \sin(\sqrt{x+1}) + \cos(x^2) \cdot \frac{d}{dx}\sin(\sqrt{x+1})$$

$$= -\sin(x^2) \cdot 2x \sin(\sqrt{x+1}) + \cos(x^2) \cdot \cos(\sqrt{x+1}) \cdot \frac{1}{2} \cdot (x+1)^{-\frac{1}{2}}$$

2. (20 points) Calculate the following limits. **Be sure to indicate if the limit is $+\infty$ or $-\infty$.**

(a) $\lim_{x \to 2} \frac{\sqrt{x+2}-2}{x-2}$

$$= \lim_{x \to 2} \frac{(\sqrt{x+2}-2)(\sqrt{x+2}+2)}{(x-2)(\sqrt{x+2}+2)}$$

$$= \lim_{x \to 2} \frac{x-2}{(x-2)(\sqrt{x+2}+2)}$$

$$= \lim_{x \to 2} \frac{1}{\sqrt{x+2}+2}$$

$$= \frac{1}{\sqrt{2+2}+2}$$

$$= \frac{1}{4}$$

(b) $\lim_{x \to -\infty} \frac{x^3+2x^2-2}{x-2x^2}$

$$= \lim_{x \to -\infty} \frac{x^3(1+\frac{2}{x}-\frac{2}{x^3})}{x^2(\frac{1}{x}-2)}$$

$$= \lim_{x \to -\infty} \underset{-\infty}{x} \cdot \frac{1+\frac{2}{x}-\frac{2}{x^3}}{\frac{1}{x}-2} \quad \text{approaches } -\frac{1}{2}$$

$$= \infty$$

(it continues in the next page)

(c) $\lim_{x \to 0} x^p \cos\left(\frac{1}{x}\right)$ with $p > 0$ (hint: use the Sandwich Theorem).

$$-1 \le \cos\left(\frac{1}{x}\right) \le 1 \quad \text{for all } x \neq 0$$

$$\Rightarrow -|x^p| \le x^p \cos\left(\frac{1}{x}\right) \le |x^p|$$

$$\lim_{x \to 0} |x^p| = \lim_{x \to 0} (-|x^p|) = 0$$

$$\Rightarrow \lim_{x \to 0} x^p \cos\left(\frac{1}{x}\right) = 0 \text{ by Sandwich thm}$$

(d) $\lim_{x \to 0} \frac{1 - \cos(2x)}{x \sin(3x)}$

$$= \lim_{x \to 0} \frac{(1-\cos 2x)(1+\cos 2x)}{x \sin(3x)(1+\cos 2x)}$$

$$= \lim_{x \to 0} \frac{1-\cos^2(2x)}{x \sin(3x)} \cdot \frac{1}{1+\cos(2x)}$$

$$= \lim_{x \to 0} \frac{\sin^2(2x)}{x \sin(3x)} \cdot \frac{1}{1+\cos(2x)}$$

$$= \lim_{x \to 0} \frac{\sin 2x}{x} \cdot \frac{\sin 2x}{\sin 3x} \cdot \frac{1}{1+\cos 2x}$$

$$= 2 \cdot \frac{2}{3} \cdot \frac{1}{2}$$

$$= \frac{2}{3}$$

$$\lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} \frac{\sin 2x}{2x} \cdot 2 = 2$$

$$\lim_{x \to 0} \frac{\sin 2x}{\sin 3x} = \lim_{x \to 0} \frac{\frac{\sin 2x}{2x} \cdot 2x}{\frac{\sin 3x}{3x} \cdot 3x}$$

$$= \frac{2}{3}$$

3. (13 points) (10 points) **Use the definition of derivative as a limit** to find the derivative of the function $f(x) = \dfrac{1}{2x-1}$. **(No partial credit if you do not use the definition.)**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{2(x+h)-1} - \frac{1}{2x-1}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2x-1-(2(x+h)-1)}{(2(x+h)-1)(2x-1)}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2x}\cancel{-1} - \cancel{2x} - 2h \cancel{+1}}{(2(x+h)-1)(2x-1)h}$$

$$= -\frac{2}{(2x-1)^2}$$

(3 points) Find the equation of the line tangent to the graph of $f(x) = \dfrac{1}{2x-1}$ at the point $x = 1$.

$$f'(1) = -\frac{2}{(2-1)^2} = -2$$

$$f(1) = \frac{1}{2-1} = 1$$

$\Rightarrow$ eqn of tangent at $x=1$ is

$$y - 1 = -2(x-1)$$

4. (18 points) Consider the rational function $f(x) = \dfrac{x^2 - x - 2}{4x^2 - 4}$.

(a) (8 points) Find all the horizontal asymptotes of the graph of $f(x)$. **Show your work.**

$$\lim_{x\to\infty} \frac{x^2-x-2}{4x^2-4} = \lim_{x\to\infty} \frac{\cancel{x^2}(1-\frac{1}{x}-\frac{2}{x^2})}{\cancel{x^2}(4-\frac{4}{x^2})} = \frac{1}{4}$$

$\Rightarrow$ f(x) is asymptotic to $y=\frac{1}{4}$ as $x\to\infty$

$$\lim_{x\to-\infty} \frac{x^2-x-2}{4x^2-4} = \lim_{x\to-\infty} \frac{\cancel{x^2}(1-\frac{1}{x}-\frac{2}{x^2})}{\cancel{x^2}(4-\frac{4}{x^2})} = \frac{1}{4}$$

$\Rightarrow$ f(x) is asymptotic to $y=\frac{1}{4}$ as $x\to-\infty$

(b) (10 points) Find all the vertical asymptotes of the graph of $f(x)$. **Show your work.**

$$f(x) = \frac{x^2-x-2}{4x^2-4} = \frac{(x+1)(x-2)}{4(x+1)(x-1)}$$

possible places for V.A: $x=-1$, $x=1$

$$\lim_{x\to-1} f(x) = \lim_{x\to-1} \frac{x-2}{4(x-1)} = \frac{-3}{-8} = \frac{3}{8}$$ $\Rightarrow$ $x=-1$ is not a V.A

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} \frac{x-2}{4(x-1)} = -\infty$$ $\Rightarrow$ $x=1$ is a V.A of f(x)

5. (16 points)

(a) Are the following statements TRUE or FALSE? If FALSE, give a counterexample.

i. A function is continuous at a whenever the left and right limits at a are equal.

False. e.g. $f(x) = \begin{cases} x & x \neq 0 \\ 1 & x = 0 \end{cases}$ at $x=0$.

f(x) may have a removable discontinuity.

ii. If a point a is not in the domain of f, then f is not continuous at a.

True.

iii. $f(x)$ and $g(x)$ are functions defined near $x = a$. If $\lim_{x \to a} f(x) = 0$, then we must have $\lim_{x \to a} f(x)g(x) = 0$.

False. e.g. $f(x) = x$. $g(x) = \frac{1}{x}$. $\lim_{x \to 0} x = 0$ but $\lim_{x \to 0} x \cdot \frac{1}{x} = 1$

(b) $f(x)$ is an odd function and $f'(1) = 1$. What can we say about $f'(-1)$?

A. $f'(-1) = -1$

B. $f'(-1) = 1$ because the graph of an odd fct is symmetric about origin.

C. There is not enough information to tell whether $f'(-1)$ exists or what $f'(-1)$ is.

(or A, C)

(c) From the following four functions, __C__ is continuous, __A__ has a point of removable discontinuity, __B__ has a point of jump discontinuity.

A. $f(x) = \dfrac{x^2 - 1}{x + 1}$

continuous at all points in its domain.

B. $f(x) = \begin{cases} \sin x & x \geq 0 \\ 1 & x < 0 \end{cases}$

C. $f(x) = \begin{cases} \cos 2x & x \geq 0 \\ \dfrac{\sin x}{x} & x < 0 \end{cases}$

continuous for all values of x

D. $f(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 1 & x = 0 \end{cases}$

6. (18 points)

(a) (8 points) Explain why the function $f(x) = |x-1|$ is not differentiable at $x = 1$.

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1} \frac{|x-1|}{x-1} \text{ does not exist}$$

since $\displaystyle \lim_{x \to 1+} \frac{|x-1|}{x-1} = \lim_{x \to 1+} \frac{x-1}{x-1} = 1$

and $\displaystyle \lim_{x \to 1-} \frac{|x-1|}{x-1} = \lim_{x \to 1-} \frac{-(x-1)}{x-1} = 1$

are not equal.

(b) (10 points) Find the value or values of the parameters a and b that make the function f differentiable at $x = 0$

$$f(x) = \begin{cases} \sin ax & x \geq 0 \\ x + b & x < 0 \end{cases}$$

① $\displaystyle \lim_{x \to 0+} f(x) = \lim_{x \to 0+} \sin(ax) = \sin(a \cdot 0) = 0$

$\displaystyle \lim_{x \to 0-} f(x) = \lim_{x \to 0-} (x+b) = b$

$f(0) = \sin(a \cdot 0) = 0$

f is continuous at $x=0 \Leftrightarrow$ the above three are equal $\Leftrightarrow b=0$

② Right derivative at $x=0$: $\displaystyle \frac{d}{dx} \sin(ax) \Big|_{x=0} = a\cos(ax) \Big|_{x=0} = a$

Left derivative at $x=0$: $\displaystyle \frac{d}{dx} (x+b) \Big|_{x=0} = 1$

$\Rightarrow f$ is diff at $x=0$ if and only if $a=1$, $b=0$.